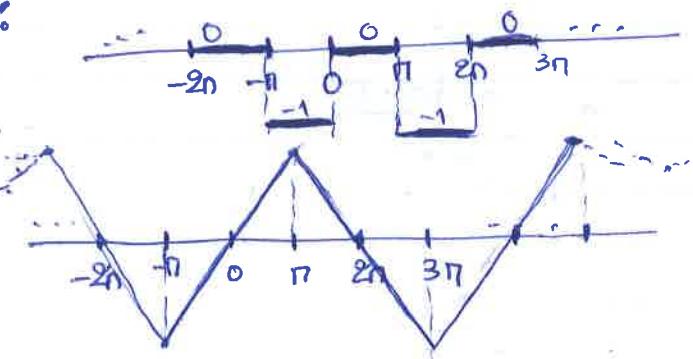


Fourier series:

→ Def: A function $f: \mathbb{R} \rightarrow \mathbb{C}$ is periodic with period L if $f(x+L) = f(x)$, $\forall x \in \mathbb{R}$.

Such a function is fully determined by the values it takes on an interval of length L (any such interval will do).

- Examples:
- $\sin x$ has period 2π .
 - $\cos x$ has period 2π .
 - $\sin 2x$ has period π ; but also 2π
↳ grows faster than x
 - $\sin(nx), \cos(nx)$ have period $\frac{2\pi}{n}$; but also 2π
 - e^{ix} has period 2π .
 - Square wave:
has period 2π .
 - Sawtooth:
has period 4π .



(2)

The square wave and sawtooth functions are very rough; and yet, because they are periodic, they can be written as sums of sines and cosines! In particular:

→ **Dirichlet's conditions :** Let $f: [-n, n] \rightarrow \mathbb{C}$,

extended to $f: \mathbb{R} \rightarrow \mathbb{C}$ $2n$ -periodically.

If $f: [-n, n] \rightarrow \mathbb{C}$ has:

- finitely many discontinuities,
- " " " minima,
- " " " maxima, and

then f can be written as a Fourier series, i.e. $\int_{-n}^n |f(x)| dx < \infty$ (we say that then $f \in L^1([-n, n])$, or f is integrable)

A Fourier series.

$$f(x) = \sum_{k \in \mathbb{Z}} c_k \cdot e^{ikx} \quad \forall x \in \mathbb{R} \text{ where } f \text{ is continuous.}$$

- ex: $\sin(\frac{1}{x}) : \mathbb{R} \rightarrow \mathbb{C}$ doesn't satisfy Dirichlet's conditions; the maximum is attained infinitely many times for x close to 0.

⚠ Notice how easy it is for $f: [-n, n] \rightarrow \mathbb{C}$

to have a Fourier series expansion! It just needs

to satisfy the 4 conditions above; it doesn't even need to be differentiable. While for a power series expansion to exist, f had to be smooth (i.e., to have infinitely many derivatives.)

(3)

A The function $f: (0, 2\pi] \rightarrow \mathbb{R}$, with $f(x) = \frac{1}{x}$ for $x \in (0, 2\pi]$,
satisfies all Dirichlet's conditions apart from
the 4-th: $\int_0^{2\pi} \frac{1}{x} dx = \infty$. (i.e., $\frac{1}{x} \notin L^1((0, 2\pi])$)

(4)

Before we discuss why we even care about Fourier series expansions, let us first see how we would find the coefficients c_k , $k \in \mathbb{Z}$, if we knew that f can indeed be written as a Fourier series:

Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be periodic, with period $2n$.

Suppose that $f(x) = \sum_{k \in \mathbb{Z}} c_k \cdot e^{ikx}$, $\forall x \in \mathbb{R}$.

Then, $c_k = \frac{1}{2n} \cdot \int_{-\pi}^{\pi} e^{-ikx} f(x) dx, \forall k \in \mathbb{Z}$

the k -th Fourier coefficient of f

Proof: A vital observation is that the functions

$e^{ikx}: \mathbb{R} \rightarrow \mathbb{C}$, over $k \in \mathbb{Z}$, are what we

call orthonormal: $\int_0^{\pi} e^{inx} \cdot e^{-imx} dx =$

$$= \begin{cases} 2\pi, & \text{if } n=m \\ 0, & \text{if } n \neq m \end{cases}$$

(because $\int_{-\pi}^{\pi} e^{ikx} dx = \begin{cases} 2\pi, & \text{if } k=0 \\ 0, & \text{if } k \neq 0 \end{cases}$).

(5)

$$\begin{aligned}
 \text{So, } \int_{-\pi}^{\pi} e^{-ikx} f(x) dx &= \int_{-\pi}^{\pi} e^{-ikx} \cdot \left(\sum_{n \in \mathbb{Z}} c_n \cdot e^{inx} \right) dx = \\
 &= \sum_{n \in \mathbb{Z}} c_n \cdot \underbrace{\int_{-\pi}^{\pi} e^{-ikx} \cdot e^{inx} dx}_{\begin{array}{l} "0 \text{ if } k \neq n, \\ \text{and } 2\pi \text{ if } k = n \end{array}} = \\
 &= c_k \cdot 2\pi \rightarrow c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx.
 \end{aligned}$$

Thus, if f can be written as a Fourier series, then this series is unique, and the coefficients are given by the above formula. See how f , a function with uncountably many values, is eventually determined fully by countably many values; its Fourier coefficients! This is just the start of why Fourier series are useful in technology, such as signal processing (remember, computers cannot deal with uncountably many inputs!)

(6)

→ We usually denote the k -th Fourier coefficient of f by $\hat{f}(k)$. Thus, If f can be written as a $\hookrightarrow_{2n\text{-periodic}}$ Fourier series, then

$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) \cdot e^{ikx}, \quad \forall k \in \mathbb{Z},$$

$$\text{where } \hat{f}(k) := \frac{1}{2n} \int_{-n}^n e^{-ikx} f(x) dx.$$



We see really here that the orthogonal functions e^{ikx} , for $k \in \mathbb{Z}$, form a basis of the space of $2n$ -periodic functions (for which Fourier decomposition exists):

In other words, they behave a bit like coordinate axes do in \mathbb{R}^n . They correspond

(or vectors) (for which Fourier expansion exists)
 each f to its coordinates $(\dots, \hat{f}(-3), \hat{f}(-2), \hat{f}(-1), \hat{f}(0), \hat{f}(1), \hat{f}(2), \dots)$,

which fully determine f ; just like the coordinates of a vector in \mathbb{R}^n fully determine the vector.

(7)

This is not a coincidence! Just like the coordinate axes are orthogonal in \mathbb{R}^n , similarly the functions $\{e^{ixk}, k \in \mathbb{Z}\}$ are orthogonal in the space of functions

$$L^2(-\pi, \pi) := \{f : [-\pi, \pi] \rightarrow \mathbb{C} \text{ with } \int_{-\pi}^{\pi} |f|^2 < +\infty\}$$

The "dot product" in this space,
(really: inner product)

such that, if $f \cdot g = 0$, then we call f and g "orthogonal",
is: $f \cdot g := \int_{-\pi}^{\pi} f \cdot \bar{g}$ conjugate.

Notice that, indeed, with this dot product:

$$e^{imx} \cdot e^{inx} = 0 \text{ if } m \neq n \text{ in } \mathbb{Z},$$

so e^{imx}, e^{inx} are orthogonal in $L^2(-\pi, \pi)$.

x runs;
 e^{imx} is
a function: $\mathbb{R} \rightarrow \mathbb{C}$.

Eventually, $L^2(-\pi, \pi)$ is a Hilbert space, which means that, with the above "dot product",

(8)

it behaves a lot like \mathbb{R}^n ; and, eventually, the orthogonal functions $\{e^{ikx} : k \in \mathbb{Z}\}$ generate $L^2(-\pi, \pi)$, just like orthogonal vectors in \mathbb{R}^n generate \mathbb{R}^n .

Unfortunately this beautiful theory is for the Spring term, but it would be good if you revisited this when you learn about inner products. This is not a triviality, it is one of the fundamental properties of Fourier series, so it deserves to be understood.



Notice that, when $f: \mathbb{R} \rightarrow \mathbb{R}$

\nearrow 2n-periodic \nearrow not \in

$$\hat{f}(-k) = \overline{\hat{f}(k)}, \quad \forall k \in \mathbb{Z}.$$

Indeed:

$$\begin{aligned} \hat{f}(k) &= \frac{1}{2\pi n} \cdot \int_{-\pi}^{\pi} e^{ikx} f(x) dx = \frac{1}{2\pi n} \cdot \int_{-\pi}^{\pi} (\overline{e^{-ikx}}) \cdot f(x) dx \\ &= \frac{1}{2\pi n} \cdot \int_{-\pi}^{\pi} (\overline{e^{-ikx}}) \cdot \overline{f(x)} dx = \frac{1}{2\pi n} \int_{-\pi}^{\pi} (\overline{e^{-ikx} f(x)}) dx = \\ &\stackrel{\text{f(x)} \in \mathbb{R}, \forall x \in \mathbb{R}}{=} \frac{1}{2\pi n} \int_{-\pi}^{\pi} \overline{e^{-ikx} f(x)} dx = \overline{\hat{f}(k)}. \end{aligned}$$

That is

$$so \hat{f}(x) = \overline{\hat{f}(x)} \quad \forall x \in \mathbb{R}$$

This is what makes the Hilbert transform ~~so useful~~ so useful in signal processing (see comment in earlier lecture).

(3)

When f has finitely many discontinuities, in principle the Fourier series of f doesn't converge at the values of f at those points. In particular, the following holds:

→ **Dirichlet's theorem:** Let $f: [-\pi, \pi] \rightarrow \mathbb{C}$

If f has:

- finitely many discontinuities,
- " " minima,
- " " maxima, and
- $\int_{-\pi}^{\pi} |f(x)| dx < +\infty$, then

$$\sum_{k \in \mathbb{Z}} \hat{f}(k) \cdot e^{ikx} = \begin{cases} f(x), & \text{if } x \text{ where } f \text{ is continuous} \\ \frac{f^+(x) + f^-(x)}{2}, & \text{if } x \text{ where } f \text{ is discontinuous} \end{cases}$$

|| by definition!

$$\lim_{N \rightarrow +\infty} \sum_{k=-N}^N \hat{f}(k) e^{ikx}$$

the midpoint
of the jump

where $\hat{f}^+(x) = \lim_{y \rightarrow x^+} f(y)$, $\hat{f}^-(x) = \lim_{y \rightarrow x^-} f(y)$.

(Notice that, if f is continuous at x , then

$$f(x) = \frac{f^+(x) + f^-(x)}{2}$$

so the Fourier series equals this

(10)

 $\mathbb{H} \times)$

Δ When $f: \mathbb{R} \rightarrow \mathbb{R}$ is $2n$ -periodic,
then $\hat{f}(-k) = \overline{\hat{f}(k)}$ (of course these coefficients
may be complex and not
real).

$$\begin{aligned} \text{Indeed, } \hat{f}(-k) &= \frac{1}{2n} \cdot \int_{-n}^n e^{ikx} f(x) dx = \\ &= \frac{1}{2n} \cdot \int_{-n}^n \overline{e^{-ikx}} \cdot \overline{f(x)} dx \quad \begin{array}{l} \text{as } f(x) = \overline{f(x)} \text{ if } x \in \mathbb{R}, \\ \text{as } f \text{ real} \end{array} \\ &= \frac{1}{2n} \cdot \int_{-n}^n \overline{e^{-ikx}} \cdot \overline{f(x)} dx = \\ &= \frac{1}{2n} \cdot \left[\int_0^n \overline{e^{-ikx} \cdot f(x)} dx \right] = \left(\frac{1}{2n} \cdot \int_{-n}^n e^{-ikx} f(x) dx \right) = \\ &= \overline{\hat{f}(k)} \end{aligned}$$

This is what makes the Hilbert transform so

useful in signal processing (see comment in earlier
lecture).

(11)

→ A $2n$ -periodic function with a Fourier series expansion can be written as a sum of (multiples of) sine functions $\sin(kx)$, $k \in \mathbb{Z}$ and cosine functions $\cos(kx)$, $k \in \mathbb{Z}$.

This is obvious by replacing each e^{ikx} with $\cos(kx) + i\sin(kx)$. Let us think a little more about what this means, though.

It means that



$$f(x) = c_0 + a_1 \cos x + a_2 \cos(2x) + a_3 \cos(3x) + \dots$$

$$\int_{-n}^n e^{-ix} f(x) dx = \int_{-n}^n f(x) dx + \dots$$

$$+ b_1 \sin x + b_2 \sin(2x) + b_3 \sin(3x) + \dots$$

If $f: \mathbb{R} \rightarrow \mathbb{C}$ $2n$ -periodic (under the Dirichlet conditions).

- Notice how here there are no terms involving kx for negative k ; naturally, we used that $\sin(-kx) = -\sin(kx)$ and $\cos(-kx) = \cos(kx)$ $\forall k \in \mathbb{Z}$.
- More importantly, we have that the coefficients $c_0, a_1, a_2, a_3, \dots, b_1, b_2, b_3, \dots$ above are real when $f: \mathbb{R} \rightarrow \mathbb{R}$.

(12)

This is not obvious from the process we followed to get $\hat{f}(k)$, as $\hat{f}(k)$ may well be complex (and not real) even when f is real (see square wave function example later).

The reason for this is that, by writing

$$\cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2} \quad \text{and} \quad \sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i},$$

we get that $\cos(kx)$ is orthogonal to $\sin(mx)$ if $m \in \mathbb{Z}$

and also orthogonal to $\cos(mx)$ if $m \neq k$ in \mathbb{Z} ,

(check it), so, just like when we found $\hat{f}(k)$, we have:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(kx) \cdot f(x) dx \quad \forall k \in \mathbb{N} \setminus \{0\}$$

$$\text{and } b_k = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \sin(kx) \cdot f(x) dx, \quad \forall k \in \mathbb{N} \setminus \{0\},$$

which are real when f takes values in \mathbb{R} !

(13)

⚠ Notice how, plugging in $\cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2}$ if $k \in \mathbb{Z}$

$$\text{and } \sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i} \text{ if } k \in \mathbb{Z}$$

into Ⓛ, we get

$$\begin{aligned} \text{Ⓐ} \Rightarrow f(x) &= c_0 + \sum_{k=1}^{+\infty} a_k \cdot \frac{e^{ikx} + e^{-ikx}}{2} + \sum_{k=1}^{+\infty} b_k \cdot \frac{e^{ikx} - e^{-ikx}}{2i} \\ &= c_0 + \sum_{k=1}^{+\infty} \underbrace{\left(\frac{a_k - ib_k}{2} \right)}_{\substack{f(0) \\ \text{for } k \geq 1}} e^{ikx} + \sum_{k=1}^{+\infty} \underbrace{\left(\frac{a_k + ib_k}{2} \right)}_{\substack{f(k), \\ f(k), \text{ for } k \leq 1}} e^{-ikx} \end{aligned}$$

So, this provides a second proof that

$$\hat{f}(-k) = \overline{\hat{f}(k)} \text{ if } k \in \mathbb{Z}, \text{ as well}$$

as the relationship between the Fourier coefficients

and the coefficients in the trigonometric expression ⚡

this gives us the relationship between the Fourier coefficients of f , and the coefficients in its expression as a sum of sine and cosine functions.

(14)

→ Examples:

- $f(x) = \boxed{e^{ix}}$: This is a periodic function with period 2π , and it is already written as a Fourier series: $\hat{f}(1) = 1$ and $\hat{f}(k) = 0, \forall k \in \mathbb{Z} \setminus \{1\}$ (because Fourier expansions are unique).
- $f(x) = \boxed{\sin x}$: This is a periodic function with period 2π , and $\sin x = \frac{e^{ix} - e^{-ix}}{2i} = \underbrace{\left(-\frac{1}{2i}\right)}_{\hat{f}(1)} e^{-ix} + \underbrace{\left(\frac{1}{2i}\right)}_{\hat{f}(1)} e^{ix}$. Notice that $\hat{f}(k) = 0 \quad \forall k \in \mathbb{Z} \setminus \{-1, 1\}$.

- Square wave function: $f(x) = \begin{cases} -1, & x \in [-\pi, 0) \\ 0, & x \in [0, \pi) \end{cases}$

periodically extended to the whole of \mathbb{R} .

Then: let $k \in \mathbb{Z}$. $\hat{f}(k) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} e^{-ikx} f(x) dx =$

$$= \frac{1}{2\pi} \cdot \int_{-\pi}^0 e^{-ikx} \cdot (-1) dx + \frac{1}{2\pi} \cdot \int_0^\pi e^{-ikx} \cdot 0 dx =$$

$$= -\frac{1}{2\pi} \cdot \int_{-\pi}^0 \frac{1}{-ik} \cdot (e^{-ikx})' dx = \frac{1}{2\pi ik} \cdot [e^{-ikx}]_{-\pi}^0 =$$

(15)

$$= \frac{1}{2\pi i k} \cdot \left(e^0 - e^{nk \cdot i} \right) = \begin{cases} 0, & \text{if } k \text{ is even} \\ \frac{1}{\pi k}, & \text{if } k \text{ is odd.} \end{cases}$$

Thus,

$$f(x) = \sum_{k \text{ odd}} \frac{1}{\pi k} \cdot e^{ikx} \quad \begin{matrix} \text{(as the Dirichlet conditions are satisfied,)} \\ \text{and } f \text{ is continuous} \end{matrix}$$

$$\text{Notice how indeed } \hat{f}(-k) = -\frac{1}{\pi k} = \frac{1}{\pi k} = \hat{f}(k),$$

+ k odd
(obvious for k even).

(15)

$$= \frac{1}{2\pi k} \cdot \left(\frac{e^0 - e^{nk \cdot i}}{i} \right) = \begin{cases} 0, & \text{if } k \text{ is even} \\ \frac{1}{\pi k}, & \text{if } k \text{ is odd.} \end{cases}$$

Thus,

$$f(x) = \sum_{k \text{ odd}} \frac{1}{\pi k} \cdot e^{ikx} \quad (\text{as the Dirichlet conditions are satisfied, and } f \text{ is continuous})$$

$$\text{Notice how indeed } \hat{f}(-k) = -\frac{1}{\pi k} = \frac{1}{\pi k} = \hat{f}(k),$$

 $\forall k \in \mathbb{Z}$

Lecture 20:

(obvious for k even).

①

Motivation for Fourier series

→ Fourier series (and Fourier transforms, in general, of non-periodic functions, which we will discuss later)

are very important in partial differential equations (PDE),

because "Fourier transforms make derivatives disappear"!

In the setting of Fourier series, this means that

diff

$$\hat{f}'(k) = ik \cdot \hat{f}(k), \quad \forall k \in \mathbb{Z}$$

→ f' exists!

In other words, if we know how to write f as a Fourier series (i.e., if we know all $\hat{f}(k), k \in \mathbb{Z}$)

then we automatically know the coefficients

for f' , and thus we know f' ! There is no need to differentiate at all! Δ f' exists and equals its Fourier series!!!

as long as we are sure that

Side comment:

(2)

As $(c_k e^{ikx})' = (ik \cdot c_k) \cdot e^{ikx}$, \star diff would be

obvious if we knew we were allowed to differentiate a Fourier series term by term. Since we don't

know that, we will prove \star diff using integration by parts; and, once \star diff is proved, it means

exactly that, IF f' exists and is equal to a Fourier series, THEN we can differentiate the Fourier series of f term by term, and the resulting series equals f' .

→ Proof of $\hat{f}'(k) = ik \cdot \hat{f}(k)$, $\forall k \in \mathbb{Z}$:

$$\hat{f}'(k) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} e^{-ikx} f'(x) dx =$$

$$= \frac{1}{2\pi} \cdot \underbrace{\left[e^{-ikx} f(x) \right]_{-\pi}^{\pi}}_{\text{II}} - \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} (e^{-ikx})' f(x) dx =$$

as both e^{-ikx} and $f(x)$
are 2π -periodic

$$= - \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} -ik \cdot e^{-ikx} f(x) dx =$$

$$= ik \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx = ik \cdot \hat{f}(k).$$

3

This was exactly Fourier's breakthrough, when he came up with Fourier series to solve the heat equation on a circle. In particular, let S^1 be the unit circle, and suppose that we have a particular temperature distribution $f: S^1 \rightarrow \mathbb{R}$ at time 0. Then, what is the temperature distribution $v(x; t)$ at time t ? I.e., at time t , what is the temperature $v(x; t)$ of each point x of S^1 , given that $v(x; 0) = f(x) \quad \forall x \in S^1$?

It is actually known that

$$\frac{\partial v}{\partial t} = \alpha \cdot \frac{\partial^2 v}{\partial x^2}(x; t) \rightarrow \text{the heat equation}$$

for α a constant that depends on the material.

But how do we solve this? Fourier noticed that, if one could write the solution as

$$v(x; t) = \sum_{k \in \mathbb{Z}} q_k(t) e^{ikx}, \text{ then, due to } \text{*diff},$$

(4)

we would have that

$$\frac{\partial v}{\partial x}(x; t) = \sum_{k \in \mathbb{Z}} ik c_k(t) \cdot e^{ikx},$$

and, again by ~~diff~~,

$$\frac{\partial^2 v}{\partial x^2}(x; t) = \sum_{k \in \mathbb{Z}} (ik)^2 c_k(t) e^{ikx} = \sum_{k \in \mathbb{Z}} (-k^2) c_k(t) e^{ikx}.$$

Moreover, $\frac{\partial v}{\partial t}(x; t) = \sum_{k \in \mathbb{Z}} c'_k(t) e^{ikx}.$

Thus, by the heat equation,

$$c'_k(t) = -k^2 a c_k(t), \quad \forall k \in \mathbb{Z}.$$

So, $\forall k \in \mathbb{Z}$, we just have a simple ODE, which

we know how to solve: $c_k(t) = e^{-k^2 at}$

$\cdot c_k(0)$.

a constant: the k -th Fourier coefficient of the temperature at time 0.

$$\text{So, } v(x; t) = \sum_{k \in \mathbb{Z}} c_k(t) e^{ikx} =$$

$$= \sum_{k \in \mathbb{Z}} e^{-k^2 at} c_k(0) \cdot e^{ikx}$$

notice how these Fourier coefficients of the solution

decay in time: the temperature

tends to being constant, $c_0(0)$ (it is only for $k=0$ that the coefficient doesn't decay in time).

the integral of the temperature along over circular wire at time 0! The temperature tends to the average temperature!

Properties of Fourier coefficients.

→ or, $f: \mathbb{R} \rightarrow \mathbb{C}$ 2π -periodic.

→ If $f: [-n, n] \rightarrow \mathbb{C}$ is differentiable, then

$$\hat{f}'(k) = ik \cdot \hat{f}(k), \quad \forall k \in \mathbb{Z}.$$

Proof: We have already proved this by integration by parts. ■

→ or, $f: \mathbb{R} \rightarrow \mathbb{C}$ 2π -periodic.

→ If $f: [-n, n] \rightarrow \mathbb{C}$ is continuously differentiable (i.e., f' exists and is continuous),

then

$$|\hat{f}'(k)| \leq \frac{M}{|k|}, \quad \forall k \neq 0, \quad \text{for some constant } M \text{ independent of } k.$$

In other words: $|\hat{f}'(k)| = O\left(\frac{1}{|k|}\right)$

⚠ If f satisfies all of Dirichlet's conditions, thus if f = its Fourier series,

then, by the preliminary test, $\hat{f}'(k) \cdot e^{ikx} \xrightarrow{|k| \rightarrow \infty} 0$,

i.e. $|\hat{f}'(k) \cdot e^{ikx}| \xrightarrow{|k| \rightarrow \infty} 0$, i.e. $\hat{f}'(k) \xrightarrow{k \rightarrow \infty} 0$. The fact, though, that $|\hat{f}'(k)| \leq \frac{M}{|k|} \quad \forall k$ implies that $\hat{f}'(k) \rightarrow 0$ even if f fits Fourier series. This generalises to f integrable (Riemann-Lebesgue Lemma).

Proof: We use the previous property: $\forall k \in \mathbb{Z}$,

$$|\hat{f}'(k)| = \left| \frac{1}{ik} \cdot \hat{f}'(k) \right| = \frac{1}{|k|} \cdot \frac{1}{2\pi} \left| \int_{-n}^n e^{-ixk} f'(x) dx \right|$$

≤ triangle inequality $\frac{1}{|k|} \cdot \frac{1}{2\pi} \cdot \int_{-n}^n |e^{-ixk} f'(x)| dx = \frac{1}{|k|} \cdot \frac{1}{2\pi} \int_{-n}^n |f'(x)| dx.$

(6)

Now, it is known that every continuous function on a closed interval has a maximum. Therefore, since f' is continuous on $[-\pi, \pi]$, it has a maximum on $[-\pi, \pi]$, so there exists some M s.t.

$$|f'(x)| \leq M, \forall x \in [-\pi, \pi].$$

$$\text{Thus, } |\hat{f}(k)| \leq \frac{1}{k} \cdot \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} M dx = \frac{M}{2\pi k}, \quad \text{for } k \in \mathbb{N}.$$

independent
of k

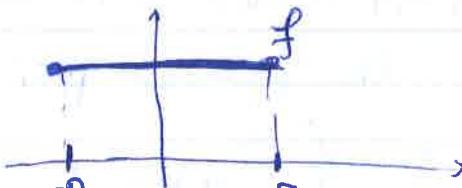
See how this implies that $\hat{f}(k) \xrightarrow[k \rightarrow \infty]{} 0$.

so, when f is continuously differentiable, and equals its Fourier series (e.g., Dirichlet conditions), it is really its coordinates $\hat{f}(k)$ for k small that really determine f ! Same holds for any f satisfying Dirichlet conditions (as then f integrable, and, by Riemann-Lebesgue lemma, $\hat{f}(k) \xrightarrow[k \rightarrow \infty]{} 0$).

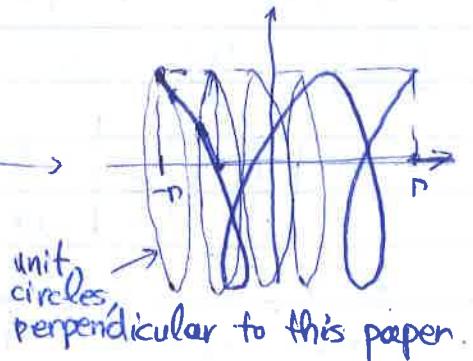


Think of $\hat{f}(k)$ as the integral of a twisted version of the graph of f , where the twisting increases as k increases (remember: e^{-ikx} just rotates

the value $f(x)$:



The integral of the twisted graph should go to 0 as the twisting



7

increases, as we have more cancellation between the twisted vectors! Hopefully this gives some intuition on why $f(k) \xrightarrow{k \rightarrow \infty} 0$, even when f is just integrable.

(7)

increases, as we have more cancellation between the twisted vectors! Hopefully this gives some intuition on why $\hat{f}(k) \xrightarrow[k \rightarrow \infty]{} 0$, even when f is just integrable.

Lecture 21

(1)

Let's take this a little further:

→ If $f: \mathbb{R} \rightarrow \mathbb{C}$ is $2n$ -periodic and N times

continuously differentiable, then, for all $k \in \mathbb{Z}$,

$$|\hat{f}(k)| \leq \frac{M}{|k|^N}, \text{ where } M \text{ is a constant}$$

i.e.: for this fixed N , independent of k (but perhaps depending on N).

$$|\hat{f}(k)| = O\left(\frac{1}{|k|^N}\right)$$

Proof: We use \hat{f}' diff again and again, N times:

$$\hat{f}'(k) = ik \cdot \hat{f}(k), \text{ if } f \text{ differentiable, } 2n\text{-periodic.}$$

Since f' is differentiable and $2n$ -periodic,

$$\hat{f}''(k) = (ik) \cdot \hat{f}'(k) = (ik)^2 \cdot \hat{f}(k).$$

Since f'' is differentiable and $2n$ -periodic,

$$\hat{f}'''(k) = (ik) \cdot \hat{f}''(k) = (ik)^3 \cdot \hat{f}(k), \text{ etc.}$$

$$\text{Eventually: } \hat{f}^{(N)}(k) = (ik)^N \cdot \hat{f}(k) \implies$$

(2)

$$\begin{aligned}
 |\hat{f}(k)| &= \frac{1}{|k|^N} \cdot \left| f^{(N)}(x) \right| = \\
 &= \frac{1}{|k|^N} \cdot \left| \frac{1}{2\pi} \cdot \int_{-n}^n e^{-ikx} f^{(N)}(x) dx \right| \leq \\
 &\leq \frac{1}{|k|^N} \cdot \frac{1}{2\pi} \cdot \int_{-n}^n |f^{(N)}(x)| dx \leq \\
 &\leq \frac{1}{|k|^N} \cdot \frac{1}{2\pi} \cdot \int_{-n}^n M dx = \\
 &= \frac{M}{|k|^N}, \quad \text{for } k \in \mathbb{Z}
 \end{aligned}$$

$|f^{(N)}|$ continuous on $[-n, n]$,
 so it takes a maximal
 value M on $[-n, n]$

\square

3

!

The above properties tell us that , the more derivatives f has, the faster its Fourier coefficients decay.
 So, smoother functions can be efficiently stored and reconstructed by keeping only their first Fourier coefficients. This is not true for functions that are rough.

!

We have just seen that , if f is differentiable many times, then its Fourier coefficients decay, with a rate that depends on the number of derivatives of f (i.e. , the level of smoothness of f).

→ Does the converse hold? i.e. , if we know that the Fourier coefficients of f decay with a particular rate, can we say anything about how smooth f is? Yes, to an extent:

(4)

Thm:

Let $f: \mathbb{R} \rightarrow \mathbb{C}$ $2n$ -periodic,with $\int_{-n}^n |f(x)|^2 dx < \infty$ (we say then
that $f \in L^2(-n, n)$,
or that f is $L^2(-n, n)$ -integrable).

If $|\hat{f}(k)| \leq \frac{M}{|k|^{N+\epsilon}}$, for some constants
 $M, \epsilon > 0$ that don't depend on k ,

then f is N times continuously differentiable.

(Δ Notice that $|\hat{f}(k)| = O\left(\frac{1}{k}\right) \Rightarrow f$ continuously differentiable;

even $|\hat{f}(k)| = O\left(\frac{1}{k^2}\right) \not\Rightarrow f$ continuously differentiable!

We need $|\hat{f}(k)| = O\left(\frac{1}{k^{2+\epsilon}}\right)$ to be sure
 that f is continuously differentiable)

(5)

We thus have:

f N times
continuously
differentiable

$$\longrightarrow |\hat{f}(k)| = O\left(\frac{1}{k^N}\right) \quad \textcircled{*}_1$$

and

$$|\hat{f}(k)| = O\left(\frac{1}{k^{N+1+\varepsilon}}\right) \xrightarrow{\text{if } L^2\text{-integrable}}$$

f N times
continuously
differentiable.

(*)₂

The mismatch in the exponents is not a surprise... Differentiability is a local property, while $\hat{f}(k)$, $k \in \mathbb{N}$, is an integral, an average; so it makes sense that, to get any local information from average behaviour, as in $\textcircled{*}_1$, we should require even more control on the average behaviour (in the case of $\textcircled{*}_2$, we require faster decay than just $O\left(\frac{1}{k^N}\right)$).

(6)

⚠ Let's take a moment to think critically about what all the above mean for differentiation.

Consider $f: \mathbb{R} \rightarrow \mathbb{C}$ 2n-periodic, s.t.

$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}, \quad \forall x \in \mathbb{R}$$

(this holds, for instance,
if f satisfies Dirichlet's
conditions and is continuous)

Are we allowed to differentiate term by term?

I.e., are we allowed to write

$$\textcircled{*} \quad f'(x) = \sum_{k \in \mathbb{Z}} (\hat{f}(k) e^{ikx})' = \sum_{k \in \mathbb{Z}} (ik) \cdot \hat{f}(k) \cdot e^{ikx},$$

$\forall x \in \mathbb{R}$?

The answer is generally no!

(while, for power series, it was true
inside the radius of convergence).

Why is the answer generally no??

(7)

- Notice that, first of all, we don't know if f' exists or not. There are many f 's that are continuous and satisfy Dirichlet's conditions, that are not differentiable. So, it is not OK to write $\textcircled{*}$.
- Let's assume f' exists. Is $\textcircled{*}$ OK?
 Notice that (if) $\hat{f}(k) = \hat{f}'(k)$, so $\textcircled{*}$ really says that f' equals its Fourier series. This is not necessarily true for f'
 that is not continuous everywhere
 (e.g.: imagine that f' has a single point of discontinuity; then we know that its Fourier series at that point will equal the midpoint of the jump, rather than the value of f').

However, if f' is continuous, then $\textcircled{*}$ may be true
 (certainly if f' satisfies Dirichlet's conditions^{and is continuous,} for instance).

But how do we check that f' is continuous?

- Well, just check whether

$$|\hat{f}(k)| \leq \frac{M}{|k|^{2+\varepsilon}}, \quad \text{for } M, \varepsilon > 0 \text{ independent of } k!$$

for all $k \in \mathbb{N}$

If yes, then f' is certainly continuous.
 (A) May be this doesn't hold, but f' is still continuous).

- If you believe that f' is not continuous,

you can prove it if you show that

it is true that

$$|\hat{f}(k)| = O\left(\frac{1}{|k|}\right)$$

(since f' continuous $\rightarrow |\hat{f}(k)| = O\left(\frac{1}{|k|}\right)$.)

- Or, to show that f' (or f) equals its Fourier series, you can try the following (easier than Dirichlet conditions)

→ Prop: Let $f: \mathbb{R} \rightarrow \mathbb{T}$ be 2π -periodic, twice continuously differentiable.

Then, $f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}, \forall x \in \mathbb{R}$

(Remember: $\sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx} := \lim_{N \rightarrow \infty} \left(\sum_{k=-N}^N \hat{f}(k) e^{iNx} \right)$).

Proof that Fourier series converges: If f'' exists, we

have $|\hat{f}(k)| \leq \frac{M}{k^2}$, for some $M > 0$ independent of k
(as f'' exists and is continuous). (3)

So, $|\hat{f}(k)e^{ikx}| = |\hat{f}(k)| \leq \frac{M}{k^2} \quad \forall k \in \mathbb{Z},$

so $\sum_{k=-\infty}^{+\infty} |\hat{f}(k)e^{ikx}|$ converges (absolutely, in fact!),

by comparison with the convergent series

$$\sum_{k=1}^{+\infty} \frac{1}{k^2}$$

(More precisely: $\sum_{k=1}^{+\infty} |\hat{f}(k)e^{ikx}|$ converges absolutely
by the above comparison,

similarly $\sum_{k=-1}^{+\infty} |\hat{f}(k)e^{ikx}|$ converges absolutely,

and thus so does $\sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ikx}$).

Notice that, by the above proof, the Fourier series
of f converges absolutely to f when f is twice
continuously differentiable. However, this is not necessarily
true for rougher functions f . And, in general,
the error $|f(x) - \sum_{k=-N}^N \hat{f}(k)e^{ikx}|$ may be hard to

calculate (unlike with power series). Thus, sometimes a more general notion of convergence of Fourier series may be more useful: L^2 -convergence.
(rather than pointwise convergence for every $x \in \mathbb{R}$):

calculate (unlike with power series). Thus, sometimes a more general notion of convergence of Fourier series may be more useful: L^2 -convergence.
(rather than pointwise convergence for every $x \in \mathbb{R}$):

Lecture 22:

→ Def:

We define $L^2(\mathbb{E}[-\pi, \pi]) := \{f: \mathbb{E}[-\pi, \pi] \rightarrow \mathbb{C}$
s.t. $\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx \right)^{1/2} < +\infty \}$.

average energy

for each $f \in L^2(\mathbb{E}[-\pi, \pi])$, we usually

denote $\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx \right)^{1/2}$ by $\|f\|_2$.

(think of this as the "length" of the function f ; just like the length of (x_1, \dots, x_n) is $(\sum |x_i|^2)^{1/2}$).

→ Theorem: Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be a 2π -periodic function in $L^2(\mathbb{E}[-\pi, \pi])$ (i.e., $\int_{-\pi}^{\pi} |f(x)|^2 dx < +\infty$).

Then $\left\| \sum_{k=-N}^{N} \hat{f}(k) e^{ikx} - f(x) \right\|_2 \xrightarrow[N \rightarrow \infty]{} 0$.

Corollaries: (i) $\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{+\infty} |\hat{f}(k)|^2$ (Parseval's identity).

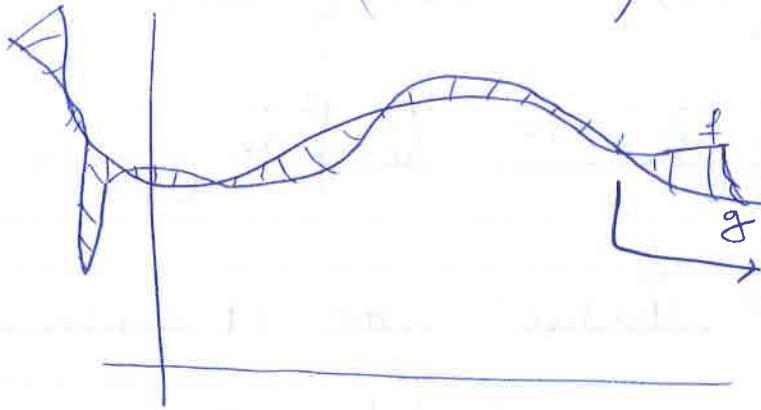
(ii) $\left\| \sum_{k=-N}^{N} \hat{f}(k) e^{ikx} - f(x) \right\|_2^2 = \sum_{k=-\infty}^{-N+1} |\hat{f}(k)|^2 + \sum_{N+1}^{+\infty} |\hat{f}(k)|^2$ (the error)

2

Let's see what this means:

① Let $f, g : [-n, n] \rightarrow \mathbb{C}$.

$$\|f-g\|_2 = \left(\frac{1}{2n} \int_{-n}^n (f-g)^2 \right)^{1/2}$$



being small means that the average energy

needed to change each $f(x)$ into $g(x)$ is small.

We think of $\|f-g\|_2$ as the distance of f and g in $L^2([-n, n])$.

$\|f-g\|_2^2$ may thus be more useful in

physics than the actual difference $f(x)-g(x) \forall x \in [-n, n]$, each of these differences may be of no importance.

⚠ $\|f-g\|_2=0$ doesn't mean that $f=g$!!! ex: Take $f \equiv g$, and change f at only one x.

But: $\|f-g\|_2=0 \iff f$ and g only differ on a set of measure 0

② There are way more functions in $L^2([-n, n])$ than those that satisfy Dirichlet's conditions.

Still though, if $f \in L^2([-n, n])$, even if

f doesn't equal its Fourier series,

i.e., if $\sum_{k=-N}^N \hat{f}(k) e^{ikx} \xrightarrow[N \rightarrow \infty]{} f(x)$

(3)

we still have that "the Fourier series of f converges to f in $L^2(-\pi, \pi)$ ",

meaning that

$$\lim_{N \rightarrow \infty} \left\| \sum_{k=-N}^N \hat{f}(k) e^{ikx} - f(x) \right\|_2 = 0,$$

i.e. $\int_{-\pi}^{\pi} \left| \sum_{k=-N}^N \hat{f}(k) e^{ikx} - f(x) \right|^2 dx \xrightarrow[N \rightarrow \infty]{} 0$.

This is a more "collective" sense of convergence than pointwise convergence; it takes the "average" behaviour over all x into account, and "glosses over" what happens at every single x .

(3)

When it comes to the "average" error

$$\left\| \sum_{k=-N}^N \hat{f}(k) e^{ikx} - f(x) \right\|_2, \text{ i.e. the error}$$

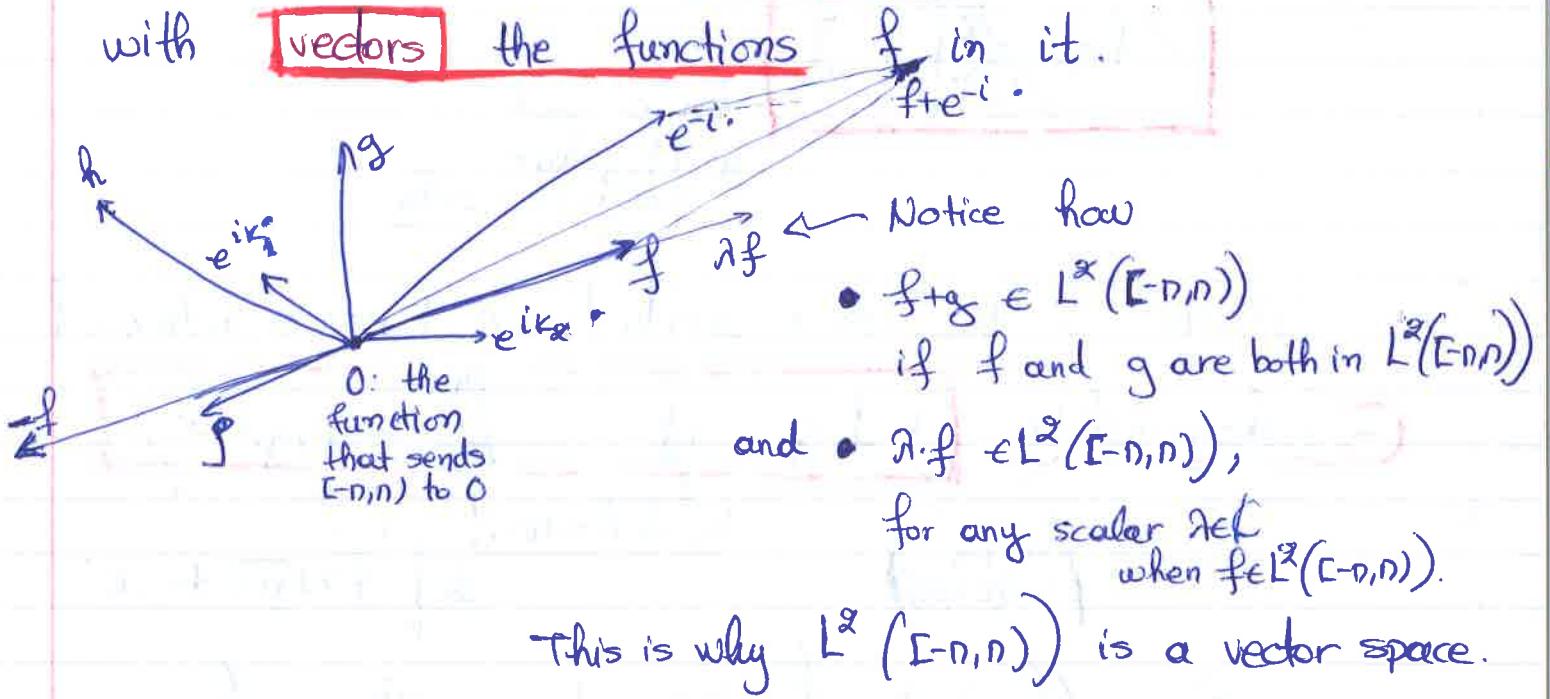
of "approximating f with $\sum_{k=-N}^N \hat{f}(k) e^{ikx}$ in $L^2(-\pi, \pi)$ ", we know

exactly what it is equal to, unlike the error

$$\sum_{k=-N}^N \hat{f}(k) e^{ikx} - f(x) \text{ for every single } x.$$

(4)

To understand the Theorem better, it really is time to start thinking of $L^2([-\pi, \pi])$ as a vector space, with vectors the functions f in it.



- In \mathbb{R}^n we have the inner (dot) product

between two vectors $(x_1, x_2, \dots, x_n) \cdot (y_1, y_2, \dots, y_n) = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$.

Thanks to this inner product, we have a notion of

- orthogonality: $(x_1, x_2, \dots, x_n) \perp (y_1, y_2, \dots, y_n)$

$$\text{if } (x_1, x_2, \dots, x_n) \cdot (y_1, y_2, \dots, y_n) = 0$$

- length: length of $(x_1, \dots, x_n) =$

$$= \sqrt{(x_1, \dots, x_n) \cdot (x_1, \dots, x_n)} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

- In $L^2(\mathbb{I}_{-\eta, \eta})$ we have the inner product $\langle \cdot, \cdot \rangle$ following
between two vectors $f, g \in L^2(\mathbb{I}_{-\eta, \eta})$:

$$\langle f, g \rangle = \frac{1}{2\eta} \int_{-\eta}^{\eta} f \cdot \bar{g}$$

this is just
 g if g takes
real values only

Thanks to this inner product, we have a notion of:

- orthogonality: $f \perp g \quad \text{if} \quad \langle f, g \rangle = 0,$
 (for f, g
in $L^2(\mathbb{I}_{-\eta, \eta})$)
 by definition!
 i.e.

$$\frac{1}{2\eta} \int_{-\eta}^{\eta} f(x) \bar{g(x)} dx = 0.$$

We call such functions f, g orthogonal.

- length: If $f \in L^2(\mathbb{I}_{-\eta, \eta})$, we define the length
of (the vector) f to be:

$$\sqrt{\langle f, f \rangle} = \left(\frac{1}{2\eta} \int_{-\eta}^{\eta} |f(x)|^2 dx \right)^{\frac{1}{2}} = \|f\|_2$$

$f(x) \cdot \bar{f(x)}$, as $f(x) \in \mathbb{C}$ & $x \in \mathbb{I}_{-\eta, \eta}$.

These notions of "orthogonality" and "length" for functions may seem artificial; however, they preserve the intuition we have from \mathbb{R}^n :

(6)

→ The Pythagorean theorem holds in $L^2(-n, n)$:

i.e., if $f, g \in L^2(-n, n)$ are orthogonal, then

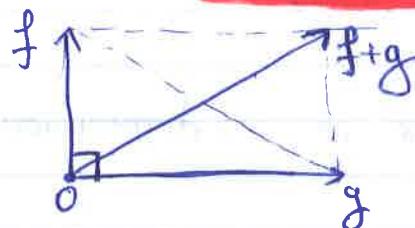
$$(\text{length of } (f+g))^2 = (\text{length of } f)^2 + (\text{length of } g)^2.$$

That is: If $f, g \in L^2(-n, n)$ are such that

$$\langle f, g \rangle = 0 \quad (\text{i.e.}, \frac{1}{2n} \int_{-n}^n f(x) \overline{g(x)} dx = 0),$$

$$\text{then } \frac{1}{2n} \int_{-n}^n |f(x) + g(x)|^2 dx = \underbrace{\frac{1}{2n} \int_{-n}^n |f(x)|^2 dx}_{\|f\|_2^2} + \underbrace{\frac{1}{2n} \int_{-n}^n |g(x)|^2 dx}_{\|g\|_2^2}$$

Proof:



$$\|f+g\|_2^2 =$$

$$= \frac{1}{2n} \int_{-n}^n |f(x) + g(x)|^2 dx = \frac{1}{2n} \int_{-n}^n (f(x) + g(x)) \cdot (\overline{f(x)} + \overline{g(x)}) dx =$$

$$= \frac{1}{2n} \int_{-n}^n (f(x) + g(x)) \cdot (\overline{f(x)} + \overline{g(x)}) dx =$$

$$= \frac{1}{2n} \int_{-n}^n \underbrace{f(x) \cdot \overline{f(x)}}_{\|f(x)\|^2} dx + \frac{1}{2n} \int_{-n}^n \underbrace{g(x) \cdot \overline{g(x)}}_{\|g(x)\|^2} dx +$$

$$+ \frac{1}{2n} \int_{-n}^n \underbrace{(g(x) \cdot \overline{f(x)})}_{\frac{1}{2n} \int_{-n}^n f(x) \overline{g(x)} dx} dx + \frac{1}{2n} \int_{-n}^n \underbrace{(f(x) \cdot \overline{g(x)})}_{\frac{1}{2n} \int_{-n}^n g(x) \overline{f(x)} dx} dx = \|f\|_2^2 + \|g\|_2^2$$

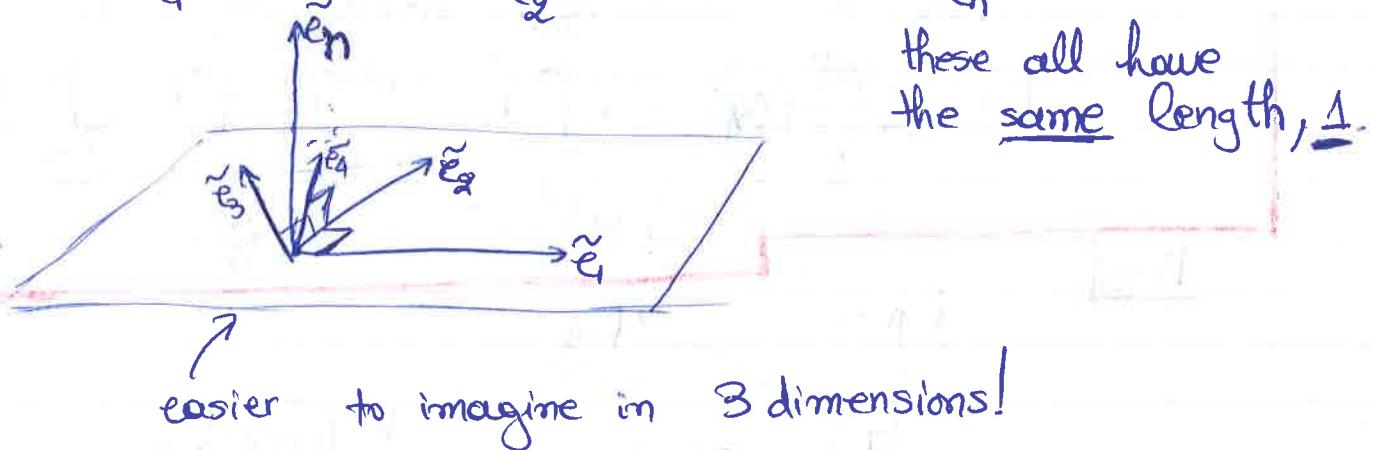
Lecture 23:

(1)

Continuing the analogy between $L^2([-\pi, \pi])$ and \mathbb{R}^n :

- In \mathbb{R}^n , we have the n pairwise orthogonal vectors

$$\underbrace{(1, 0, \dots, 0)}_{\tilde{e}_1}, \underbrace{(0, 1, 0, \dots, 0)}_{\tilde{e}_2}, \dots, \underbrace{(0, \dots, 0, 1)}_{\tilde{e}_n};$$



easier to imagine in 3 dimensions!

And, no matter what vector $\vec{x} = (x_1, x_2, \dots, x_n)$ we pick in \mathbb{R}^n , we have

$$\vec{x} = \underbrace{x_1 \tilde{e}_1 + x_2 \tilde{e}_2 + \dots + x_n \tilde{e}_n}_{\text{a vector in direction } \tilde{e}_i \text{ with different length}} , \text{ a sum of dilations of the } \tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n.$$

this is the same as saying that the distance $|\vec{x} - (x_1 \tilde{e}_1 + \dots + x_n \tilde{e}_n)|$ of \vec{x} from $x_1 \tilde{e}_1 + \dots + x_n \tilde{e}_n$ is 0.

Actually, we can't find any other $g \in L^2(\mathbb{E}_{-n,n})$ perpendicular to the e_k 's! We say that the e_k 's form a basis of $L^2(\mathbb{E}_{-n,n})$ (they are its essential building blocks). 2

- In $L^2(\mathbb{E}_{-n,n})$ we have the infinitely many pairwise orthogonal vectors $e_k = e^{ikx}$, $k \in \mathbb{Z}$, that all have the same length, 1.

Indeed: for $k \neq m$ in \mathbb{Z} :

$$\langle e^{ikx}, e^{imx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} \cdot \overline{e^{imx}} dx =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-imx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i \cdot (k-m)x} dx = 0, \text{ since } m \neq k.$$

While (length of e^{ikx}) =

$$= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{ikx}|^2 dx \right)^{1/2} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} 1 dx \right)^{1/2} = 1, \quad \forall k \in \mathbb{Z}$$

$$e^{-ikx} \cdot \overline{e^{ikx}} = e^{-ikx} \cdot e^{ikx} = 1$$

And the theorem actually tells us that, no matter what vector $f \in L^2(\mathbb{E}_{-n,n})$ we pick, we have

*
$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx} \rightarrow \text{sum of dilations of the "coordinate" vectors } e^{ikx}$$

in the $L^2(\mathbb{E}_{-n,n})$ sense, not pointwise!

I.e.: $\lim_{N \rightarrow \infty} \left\| \sum_{k=-N}^N \hat{f}(k) e^{ikx} - f(x) \right\|_2 = 0$, which means that the partial sums of the Fourier series of f converge to f in $L^2(\mathbb{E}_{-n,n})$ (but may diverge pointwise).

(3)

Moreover, Parseval's identity (a Corollary of the theorem) tells us what we have been discussing all along:

If $f \in L^2(-n, n)$, then, not only

can we express $f = (\dots, \hat{f}(-3), \hat{f}(-2), \hat{f}(-1), \hat{f}(0), \hat{f}(1), \hat{f}(2), \dots)$

(by ④, in $L^2(-n, n)$) but not necessarily pointwise,

but

$\boxed{\text{(the length of } f \text{ in } L^2(-n, n))}$

=

$\boxed{\text{(the square root of the sum of the squares)} \\ \text{of these coordinates,)}}$

$$\text{i.e. } \|f\|_2 = \left(\sum_{k=-\infty}^{+\infty} |\hat{f}(k)|^2 \right)^{1/2};$$

this is a direct generalisation of what happens in \mathbb{R}^n !

Let us now see a sketch of the proof of the Theorem, so that we understand why it makes any sense. first, we need some basic observations:

(4)

→ Observations:

(1) $\forall k \in \mathbb{Z}, \hat{f}(k) = \frac{1}{2n} \cdot \int_{-n}^n e^{-ikx} f(x) dx = \frac{1}{2n} \cdot \int_{-n}^n f(x) \cdot \overline{e^{ikx}} dx =$

$$= \underline{\underline{\langle f, e^{ik \cdot} \rangle}}$$

(2) Let $k_1, k_2, \dots, k_m \in \mathbb{Z}$, pairwise distinct.

Then, the coordinate vectors $e^{ik_1 \cdot}, e^{ik_2 \cdot}, \dots, e^{ik_m \cdot}$
are pairwise orthogonal, thus it makes sense that,

as in \mathbb{R}^m ,

$$\left\| \underbrace{\lambda_1 e^{ik_1 \cdot} + \lambda_2 e^{ik_2 \cdot} + \dots + \lambda_m e^{ik_m \cdot}}_{\text{length of } "(\lambda_1, \lambda_2, \dots, \lambda_m)" \text{ in } L^2([-n, n])} \right\|_2 = \left(\sum_{i=1}^m \lambda_i^2 \right)^{1/2}.$$

$$= \lambda_1 e^{ik_1 \cdot} + \dots + \lambda_m e^{ik_m \cdot}$$

Indeed, this holds:

$$\begin{aligned} \left\| \lambda_1 e^{ik_1 \cdot} + \lambda_2 e^{ik_2 \cdot} + \dots + \lambda_m e^{ik_m \cdot} \right\|_2^2 &= \langle \lambda_1 e^{ik_1 \cdot} + \dots + \lambda_m e^{ik_m \cdot}, \lambda_1 e^{ik_1 \cdot} + \dots + \lambda_m e^{ik_m \cdot} \rangle \\ &= \left\langle \sum_{i=1}^m \lambda_i e^{ik_i \cdot}, \sum_{j=1}^m \lambda_j e^{ik_j \cdot} \right\rangle = \sum_{i=1}^m \sum_{j=1}^m \lambda_i \bar{\lambda}_j \cdot \underbrace{\langle e^{ik_i \cdot}, e^{ik_j \cdot} \rangle}_{\begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}} \\ &= \sum_{i=1}^m |\lambda_i|^2 \end{aligned}$$

→ **SKETCH** OF PROOF of Theorem:

→ Step 1: $\forall \text{NeZ}, \left(f - \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right) \perp \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot}$

⚠ Not a surprise! We want to show really that,

in $L^2(-n, n)$, $f = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ik\cdot}$

coordinates coordinate vectors

We should thus eventually have:

$$f = \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} + \sum_{\text{rest of } k \in \mathbb{Z}} \hat{f}(k) e^{ik\cdot} = f - \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot}$$

sum of dilations of $e^{ik\cdot}$ sum of dilations of $e^{ik\cdot}$ perpendicular to these

so these two should be orthogonal

Indeed:

$$\left\langle f - \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot}, \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right\rangle =$$

$$= \left\langle f, \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right\rangle - \left\langle \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot}, \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right\rangle =$$

$$= \sum_{k=-N}^N \overline{\hat{f}(k)} \cdot \underbrace{\left\langle f, e^{ik\cdot} \right\rangle}_{\hat{f}(k)}, \text{ by Obs. ①} - \underbrace{\left\| \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right\|_2^2}_{\sum_{k=-N}^N |\hat{f}(k)|^2, \text{ by Obs. ②}} =$$

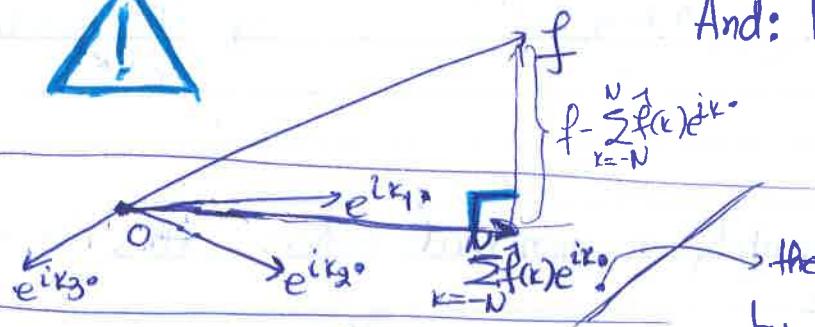
$$= \sum_{k=-N}^N |\hat{f}(k)|^2 - \sum_{k=-N}^N |\hat{f}(k)|^2 = 0.$$

6

the space generated by the orthogonal vectors
 $e^{i0^\circ}, e^{i1^\circ}, e^{-i1^\circ}, e^{i2^\circ}, e^{-i2^\circ}, \dots, e^{i(N-1)^\circ}, e^{-i(N-1)^\circ}$
 (a $(2N+1)$ -dim space)
 Functions in $L^2([-n, n])$

Pythagorean thm

So, the picture is this :



$$\text{And: } \|f\|_2^2 = \left\| f - \sum_{k=-N}^N \hat{f}(k) e^{ik*} \right\|_2^2 + \left\| \sum_{k=-N}^N \hat{f}(k) e^{ik*} \right\|_2^2$$

$\left\| \sum_{k=-N}^N \hat{f}(k) e^{ik*} \right\|_2^2 = \left(\sum_{k=-N}^N |\hat{f}(k)|^2 \right)$.

the space generated by $e^{ik*}, k \in \{-N, \dots, N\}$

Step 2: In $L^2([-n, n])$, $\sum_{k=-N}^N \hat{f}(k) e^{ik*}$ is the closest element to f , amongst all elements

in the space generated by $e^{ik*}, k \in \{-N, \dots, N\}$

(as one would expect from an orthogonal projection!)

Indeed, let $\sum_{k=-N}^N \alpha_k e^{ik*}$ be an arbitrary such

element. Then:

$$\begin{aligned} \underset{\substack{\downarrow \\ \text{in } L^2([-n, n])}}{\text{dist}} \left(f, \sum_{k=-N}^N \alpha_k e^{ik*} \right)^2 &= \left\| f - \sum_{k=-N}^N \alpha_k e^{ik*} \right\|_2^2 = \\ &= \left\| f - \sum_{k=-N}^N \hat{f}(k) e^{ik*} + \sum_{k=-N}^N \hat{f}(k) e^{ik*} - \sum_{k=-N}^N \alpha_k e^{ik*} \right\|_2^2 \end{aligned}$$

$$= \left\| f - \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right\|_2^2 + \underbrace{\left\| \sum_{k=-N}^N (\hat{f}(k) - \hat{a}_k) e^{ik\cdot} \right\|_2^2}_{\sum_{k=-N}^N |\hat{f}(k) - \hat{a}_k|^2}$$

(7)

Pythagorean theorem, as

$$\left(f - \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right) \perp \sum_{k=-N}^N (\hat{f}(k) - \hat{a}_k) e^{ik\cdot}$$

by orthogonality of the functions $e^{ik\cdot}$

So, the above quantity is minimal when this is 0,
i.e. when $\hat{a}_k = \hat{f}(k)$, $\forall k \in \{-N, \dots, N\}$.

Step 3:

For any approximation error $\epsilon > 0$,

if

N is large enough,

$$\text{element } g_N = \sum_{k=-N}^N \hat{a}_k e^{ik\cdot}$$

deep theorem, not obvious

I can find an

(an element in the space generated by the $e^{ik\cdot}, k \in \{-N, \dots, N\}$)

such that $\|f - g_N\|_2 < \epsilon$ (I can do this for N from some natural number onwards)

So, these subspaces get closer and closer to f , as N gets larger!

Since we know that $\sum_{k=-N}^N \hat{f}(k) e^{ik\cdot}$ is the closest element to f in this space, we have

(8)

that $\left\| f - \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right\|_2 \leq \|f-g\| < \epsilon$, for all large N .

Since $\epsilon > 0$ was arbitrary, we have

$$\boxed{\left\| f - \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right\|_2 \xrightarrow[N \rightarrow \infty]{} 0.} \quad (*)$$

This proves the Theorem. The corollaries are simple:

By $(*)$, $\left\| \sum_{k=-N}^{2N} \hat{f}(k) e^{ik\cdot} \right\|_2^2 \xrightarrow[N \rightarrow \infty]{} \|f\|_2^2$,

even better, by
the $(*)$ in p. 6!

$\left\| \sum_{k=-N}^{2N} \hat{f}(k) e^{ik\cdot} \right\|_2^2 \xrightarrow[N \rightarrow \infty]{} \|f\|_2^2$ (Obs. ①)

thus $\|f\|_2^2 = \sum_{k=-\infty}^{+\infty} |\hat{f}(k)|^2$ (Parseval's identity).

Since, by Step 4, $\left(f - \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right) \perp \sum_{k=N+1}^{\infty} \hat{f}(k) e^{ik\cdot}$, and

$f = \left(f - \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right) + \left(\sum_{k=N+1}^{\infty} \hat{f}(k) e^{ik\cdot} \right)$, the Pythagorean

Theorem gives:

$$\begin{aligned} \|f\|_2^2 &= \left\| f - \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right\|_2^2 + \left\| \sum_{k=N+1}^{\infty} \hat{f}(k) e^{ik\cdot} \right\|_2^2 \\ \rightarrow \quad \left\| f - \sum_{k=-N}^N \hat{f}(k) e^{ik\cdot} \right\|_2^2 &= \|f\|_2^2 - \left\| \sum_{k=N+1}^{\infty} \hat{f}(k) e^{ik\cdot} \right\|_2^2 \\ &= \sum_{k=-\infty}^{+\infty} |\hat{f}(k)|^2 - \sum_{k=-N}^N |\hat{f}(k)|^2 = \sum_{k=-\infty}^{-N-1} |\hat{f}(k)|^2 + \sum_{k=N+1}^{+\infty} |\hat{f}(k)|^2. \end{aligned}$$

(9)

Above all, remember Parseval's identity:

Parseval's identity: Let $f \in L^2(-n, n)$.

$$\text{Then, } \frac{1}{2n} \int_{-n}^n |f(x)|^2 dx = \sum_{k=-\infty}^{+\infty} |\hat{f}(k)|^2. \quad (= \sum_{k=-\infty}^{+\infty} \frac{1}{2n} \int_{-n}^n |\hat{f}(k)e^{ikx}|^2 dx)$$

Remember: $f = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ikx}$ in $L^2(-n, n)$. If f was a wave, this would mean that f is a sum of harmonics. And Parseval's identity says that the average energy of a wave is the sum of the energies of each harmonic.

See Example 1, p. 376 for a physics application.

See Example 2, p. 376, to see how to apply this identity to evaluate sums of series.

See p. 372-377 for discussion.

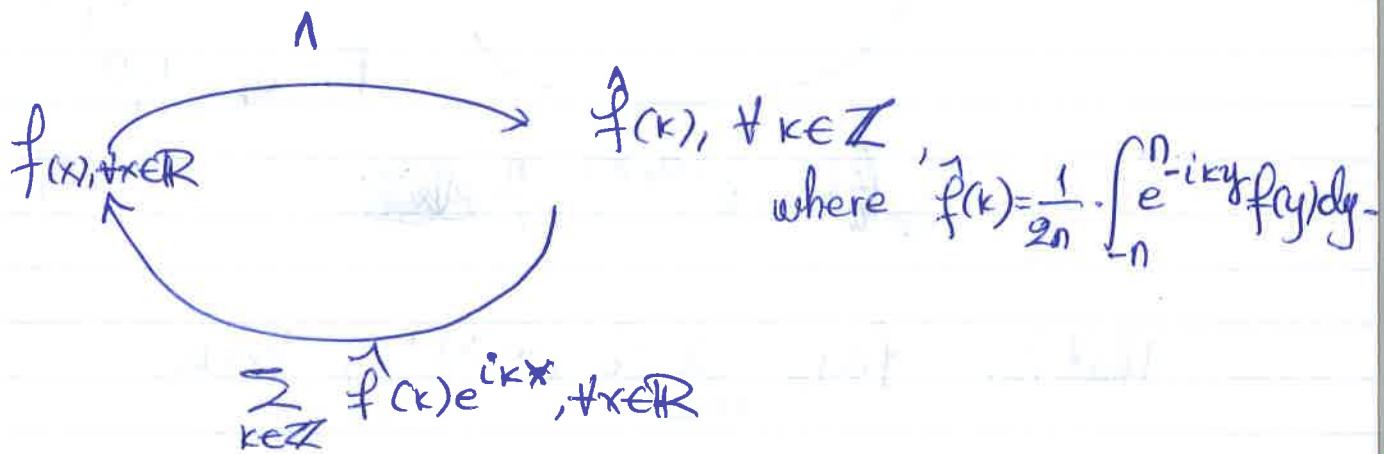
Lecture 25.

①

The Fourier transform

When we discussed Fourier series, we saw that, for appropriately "nice" functions f , we have

two processes that turn out to be inverses of each other:



This works for f $2n$ -periodic. And recall that,

if f is $2L$ -periodic, and appropriately "nice"

(e.g., satisfies Dirichlet's conditions and is continuous),

$$\text{then } f(x) = \sum_{k \in \mathbb{Z}} c_k e^{i \alpha_k x} \cdot \frac{\pi}{L}, \quad x \in \mathbb{R},$$

$$\text{where } \alpha_k = \frac{k\pi}{L} \quad \text{and} \quad c_k = \frac{1}{2\pi} \cdot \int_{-L}^L e^{-i \alpha_k y} f(y) dy$$

(2)

Therefore, for f $2L$ -periodic, the corresponding two processes are the following:

$$f(x), \forall x \in \mathbb{R}$$

$$\sum_{k \in \mathbb{Z}} (c_k \cdot e^{i\alpha_k x}) \cdot \frac{\pi}{L}$$

$$c_k, \forall k \in \mathbb{Z},$$

$$\text{where } c_k = \frac{1}{2\pi} \cdot \int_{-L}^L e^{-iy\alpha_k} f(y) dy,$$

$$\text{for } \alpha_k = \frac{k\pi}{L}$$

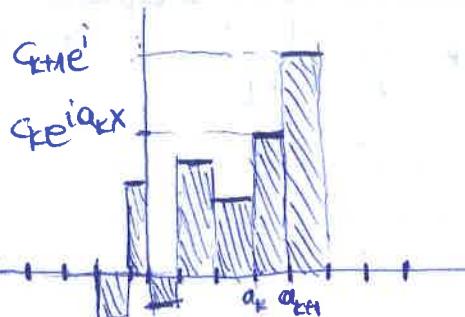
for each α_k
we have a
corresponding c_k

That is, $f(x) = \sum_{k \in \mathbb{Z}} (c_k \cdot e^{i\alpha_k x}) \left(\frac{\pi}{L} \right), \forall x \in \mathbb{R}$.

Notice that $\alpha_{k+1} - \alpha_k = \frac{(k+1)\pi}{L} - \frac{k\pi}{L} = \frac{\pi}{L}$

thus $f(x) = \sum_{k \in \mathbb{Z}} (c_k \cdot e^{i\alpha_k x}) \cdot \underbrace{(\alpha_{k+1} - \alpha_k)}_{\text{length of interval } [\alpha_k, \alpha_{k+1}]}$, $\forall x \in \mathbb{R}$

In other words, for each fixed $x \in \mathbb{R}$, $f(x)$ is just an integral: a sum of sizes of thin parallelograms:



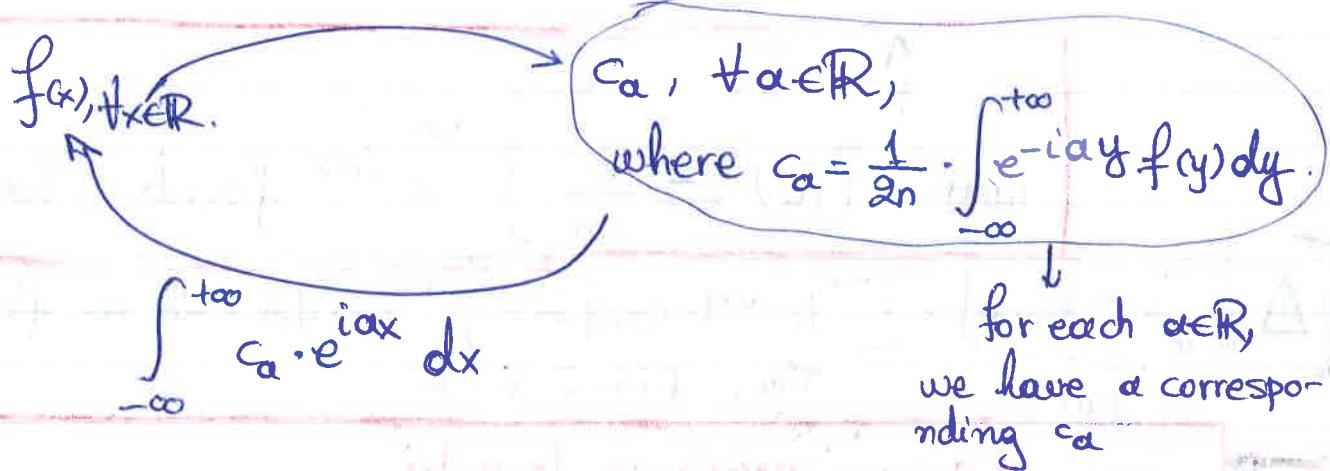
Each parallelogram has base length $\frac{\pi}{L}$, which shrinks to 0 as the period $2L \rightarrow +\infty$.

(3)

Similarly, as $2L \rightarrow +\infty$, the points $\alpha_k, k \in \mathbb{Z}$, which are $\frac{\pi}{L}$ - apart, cover more and more of \mathbb{R} .

Thus, we can heuristically believe that, when

$f: \mathbb{R} \rightarrow \mathbb{C}$ is not periodic, which is technically the case $2L = +\infty$, the corresponding two processes are the following:



In fact, the above heuristic argument can be made rigorous, and proved properly for quite general $f: \mathbb{R} \rightarrow \mathbb{R}$, that don't have to be periodic:

(4)

→ Def: We define

$$L^1(\mathbb{R}) := \left\{ f: \mathbb{R} \rightarrow \mathbb{C}, \text{ with } \int_{-\infty}^{+\infty} |f(x)| dx < +\infty \right\}.$$

→ Def: (Fourier transform):

Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be in $L^1(\mathbb{R})$.

We define the Fourier transform \hat{f} of f to be the function

$$\hat{f}: \mathbb{R} \rightarrow \mathbb{C}, \text{ with } \hat{f}(a) := \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{-iax} f(x) dx, \forall a \in \mathbb{R}.$$

[⚠] $\left| \int_{-\infty}^{+\infty} e^{-iax} f(x) dx \right| \leq \int_{-\infty}^{+\infty} |e^{-iax}| \cdot |f(x)| dx = \int_{-\infty}^{+\infty} |f(x)| dx < +\infty, \text{ as } f \in L^1(\mathbb{R}). \right]$
 Thus, $\hat{f}(a) \in \mathbb{C}, \forall a \in \mathbb{R}$.

→ Thm: Fourier inversion formula:

Let $f \in L^1(\mathbb{R})$, with $\hat{f} \in L^1(\mathbb{R})$ as well. Then:

$$f(x) = \int_{-\infty}^{+\infty} \hat{f}(a) e^{iax} da, \quad \text{for almost every } x$$

(Just know that when f is ^{on any finite interval} continuous and satisfies Dirichlet's conditions, equality holds for all x .)

(5)



See how this is a direct generalisation of
what happens for $2L$ -periodic functions, to the
 case where $2L = \infty$.

Again we can express f as a Fourier "series" —
 it's just that now the "series" has a term
 $\hat{f}(\alpha) \cdot e^{i\alpha x}$ corresponding to each $\alpha \in \mathbb{R}$, rather
 than just a discrete.

This really means that we need much more information to reconstruct a non-periodic function, than we'd need to reconstruct a periodic one. We need averages of many more twisted graphs.

→ We see that, for all $\alpha \in \mathbb{R}$,

$$f(-x) = \int_{-\infty}^{+\infty} \hat{f}(\alpha) e^{i\alpha(-x)} d\alpha = \int_{-\infty}^{+\infty} e^{-ix\alpha} \hat{f}(\alpha) d\alpha =$$

$$= 2\pi \cdot \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{-ix\alpha} \hat{f}(\alpha) d\alpha = 2\pi \cdot \hat{f}(x).$$

In other words: The Fourier transform applied to f twice gives us pretty much f ! :

(6)

both continuous, satisfying
Dirichlet's conditions on any
finite interval

→ Thm: Let $f \in L^1(\mathbb{R})$, such that $\hat{f} \in L^1(\mathbb{R})$ too.

Then : $\hat{f}(x) = \frac{1}{2\pi} \cdot f(-x), \forall x \in \mathbb{R}$.

6

f continuous,
satisfying Dirichlet's
conditions on any finite interval

→ Thm: Let $f \in L^1(\mathbb{R})$

actually,
true
when
 $f \in L^2(\mathbb{R})$,
for a.e. $x \in \mathbb{R}$.

Then: $\hat{f}(x) = \frac{1}{2\pi} \cdot f(-x), \forall x \in \mathbb{R}$.

Lecture 26

1

When f was $2n$ -periodic, Parseval's identity told us that

$$\frac{1}{2n} \int_{-n}^n |f(x)|^2 dx = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2.$$

This generalises to all $f \in L^1(\mathbb{R})$:

→ Thm: (Plancherel's theorem):

Let $f \in L^1(\mathbb{R})$. Then,

actually,
true $\forall f \in L^1(\mathbb{R})$
But then it's a
long story to define \hat{f} .

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |\hat{f}(a)|^2 da.$$

the (continuous)
Fourier transform of f .

This implies that $\mathcal{F}: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$,

i.e. the Fourier transform takes functions in $L^2(\mathbb{R})$ to functions in $L^2(\mathbb{R})$ again. Actually: it is a 1-1 and onto function, and practically an isometry, by Plancherel!

↓
the Fourier transform
preserves energy!

This follows from
this generalised version
(for $f=g$).

(2)

→ Thm: (Fourier transform duality,
i.e. generalised Plancherel):

Let $f, g \in L^1(\mathbb{R})$. Then,

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) \overline{g(x)} dx = \int_{-\infty}^{+\infty} \hat{f}(a) \overline{\hat{g}(a)} da.$$

Proof:

$$\begin{aligned}
 & \int_{-\infty}^{+\infty} \hat{f}(a) \cdot \overline{\hat{g}(a)} da = \hat{g}(a) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iax} g(x) dx \\
 &= \int_{a=-\infty}^{+\infty} \hat{f}(a) \cdot \frac{1}{2\pi} \cdot \overline{\int_{x=-\infty}^{+\infty} e^{-idx} g(x) dx} da = \\
 &= \frac{1}{2\pi} \int_{a=-\infty}^{+\infty} \hat{f}(a) \cdot \int_{x=-\infty}^{+\infty} e^{-iax} \cdot \overline{g(x)} dx da = \\
 &= \frac{1}{2\pi} \cdot \int_{a=-\infty}^{+\infty} \int_{x=-\infty}^{+\infty} \hat{f}(a) \cdot e^{iax} \overline{g(x)} dx da = \\
 &= \frac{1}{2\pi} \cdot \int_{x=-\infty}^{+\infty} \overline{g(x)} \cdot \left(\int_{a=-\infty}^{+\infty} \hat{f}(a) e^{iax} da \right) dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) \overline{g(x)} dx
 \end{aligned}$$

↓ Fourier inversion

(3)

We have so far adopted the same notation

$\hat{}$, to denote the action that takes a periodic function to its Fourier coefficients, as well as the action that takes f to its Fourier transform.

We would now like to see if there is a connection between the two, so we will use a different notation, \tilde{f} , for the Fourier transform, to avoid confusion. In particular, we know:

$$\begin{array}{ccc} f & \xrightarrow{\hat{}} & \underbrace{\tilde{f}(k), k \in \mathbb{Z}}_{\text{a sequence}}, \text{ and} \\ \text{2n-periodic} & & f(x) = \sum_{k \in \mathbb{Z}} \tilde{f}(k) e^{ikx} \\ & & \text{if } x \in \mathbb{R}, \text{ when } f \text{ is well-behaved.} \end{array}$$

So, this $\hat{}$ takes functions to sequences.

$$\begin{array}{ccc} f & \xrightarrow{\tilde{}} & \underbrace{\tilde{f}(f)}_{\text{a function}} : \mathbb{R} \rightarrow \mathbb{C}, \\ \text{in } L^1(\mathbb{R}) & \text{(Fourier transform)} & \text{with } \tilde{f}(f)(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} f(x) dx, \omega \in \mathbb{R} \end{array}$$

much more info is needed to code general f than f periodic!

\tilde{f} takes functions to functions

and $f(x) = \inf(f(f))(-x)$ $\forall x \in \mathbb{R}$, when f is well-behaved (i.e., $f(x) = \inf(f(-x))$, with previous notation).

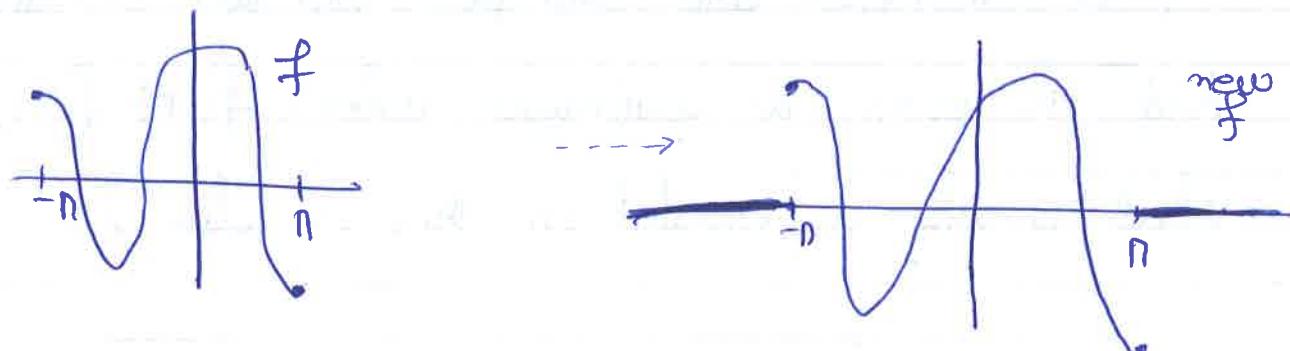
(4)

Let us start with a simple observation:

Let $f: [-n, n] \rightarrow \mathbb{C}$, $f \in L^2([-n, n])$.

Then, $\hat{f}(k) = \tilde{f}(f)(k)$ $\forall k \in \mathbb{Z}$,

Where by $\tilde{f}(f)$ we mean the (continuous) Fourier transform of $f: \mathbb{R} \rightarrow \mathbb{C}$, that is just the original f , extended to be 0 outside $[-n, n]$:



$$\text{Indeed, } \hat{f}(k) = \frac{1}{2n} \cdot \int_{-n}^n f(x) e^{-ikx} dx = \\ = \frac{1}{2n} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx = \frac{1}{2n} \cdot \tilde{f}(f)(k).$$

Since $f \in L^2([-n, n])$, we know (by Carleson) that it equals its Fourier series for almost every $x \in [-n, n]$

(5)

$$\text{so } f(x) = \sum_{k \in \mathbb{Z}} \underline{F(f)(k)} e^{ikx}, \text{ for almost every } x \in [-\pi, \pi].$$

(⚠ If we want to be sure that $f(x) = \sum_{k \in \mathbb{Z}} F(f)(k) e^{ikx}$ for all $x \in [-\pi, \pi]$, we should make sure that, when f is extended $2n$ -periodically on the whole of \mathbb{R} , then f is continuous and satisfies Dirichlet's conditions. And it won't be continuous unless $f(-n) = f(n)$, which is why we insisted on this in class).

More generally:

Let $f: [-L, L] \rightarrow \mathbb{C}$, $f \in L^2([-n, n])$.

Then,

$$f(x) = \frac{\pi}{L} \sum_{k \in \mathbb{Z}} F(f)\left(k \cdot \frac{\pi}{L}\right) e^{(k \frac{\pi}{L}) \cdot x},$$

for every $x \in [-L, L]$
if f satisfies
DC, etc

for almost every
 $x \in [-L, L]$,

where here we have extended f to be 0 outside $[-L, L]$.

(6)

Proof: We know (proved by Carleson) for $f \in L^2([-L, L])$

that $f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) \cdot e^{ikx}$ for almost every $x \in [-L, L]$,

$$\text{where now } \hat{f}(k) = \frac{1}{2L} \cdot \int_{-L}^L e^{-ik\frac{\pi}{L}x} f(x) dx =$$

$$= \frac{\pi}{L} \cdot \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{-ik\frac{\pi}{L}x} f(x) dx =$$

↓
the extended f

$$= \frac{\pi}{L} \cdot F(f)\left(\frac{k\pi}{L}\right), \quad k \in \mathbb{Z}.$$

⚠ (Again, we have equality $\forall x \in \mathbb{R}$ if f is continuous and satisfies Dirichlet's conditions, once extended $2L$ -periodically on the whole of \mathbb{R} ; and, for this periodic function to be continuous, we need $f(-L) = f(L)$).



This is all pretty boring. But, combined with the Fourier inversion formula (twice), it gives something magic :

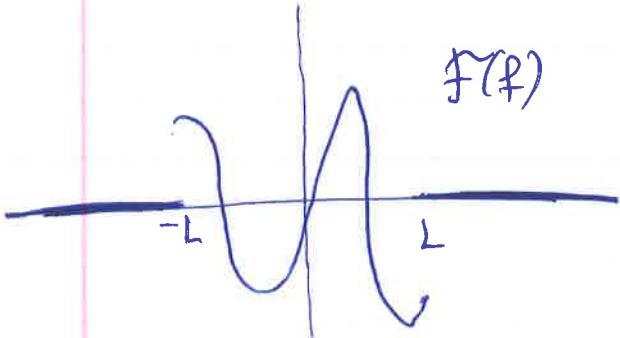
(7)

Nyquist's sampling theorem:

$f \in L^1(\mathbb{R})$, f satisfies DC
on every finite interval.

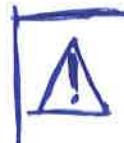
Let $f \in L^1(\mathbb{R})$, continuous, satisfying Dirichlet's conditions
(and thus $\hat{f}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) dt$)

IF the Fourier transform $\hat{f}(f)$ of f has the
property that it is 0 outside some interval $[-L, L]$,
(i.e. $\hat{f}(f)(x) = 0 \forall x \notin [-L, L]$),



then f is fully determined by the discrete samples

$$\left\{ f\left(\frac{k\pi}{L}\right) : k \in \mathbb{Z} \right\}$$



In the proof that follows, we will need
to know that $\hat{F}(\hat{f})(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(-x-t) dt$

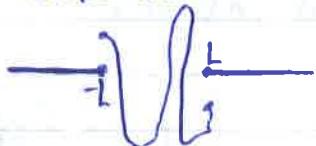
for all x in this discrete sample, so

that's why we impose Dirichlet's conditions.

Proof: $f \in L^2(\mathbb{R}) \xrightarrow{\text{Plancherel}} \tilde{f}(f) \in L^2(\mathbb{R})$

by theorem

on p. 5,
applied for $\tilde{f}(f)$,
which looks like
this



$$\tilde{f}(f)(x) = \frac{\pi}{L} \cdot \sum_{k \in \mathbb{Z}} \tilde{f}(f)(\frac{k\pi}{L}) e^{ik\frac{\pi}{L}x},$$

for almost every $x \in [-L, L]$.

(and $\tilde{f}(f)(x) = 0 \nexists x \notin [-L, L]$,
by assumption).

By the Fourier inversion formula:

$$\tilde{f}(F(f))\left(\frac{k\pi}{L}\right) = \frac{1}{2L} \cdot f\left(-\frac{k\pi}{L}\right), \forall k \in \mathbb{Z}. \quad \oplus$$

(since $F(F(f))(x) = \frac{1}{2L} f(-x)$, $\forall x \in \mathbb{R}$).

$$\text{So, } \tilde{f}(f)(x) = \frac{1}{2L} \cdot \sum_{k \in \mathbb{Z}} f\left(-\frac{k\pi}{L}\right) e^{ik\frac{\pi}{L}x} =$$

$$= \frac{1}{2L} \cdot \sum_{k \in \mathbb{Z}} f\left(\frac{k\pi}{L}\right) e^{-ik\frac{\pi}{L}x}, \text{ for almost every } x \in \mathbb{R}.$$

So, since we know $\tilde{f}(f)$, we know also f ,
by Fourier inversion ($f(x) = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} f(t) e^{itx} dt$). (we know $f(x) = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \tilde{f}(f)(y) e^{iyx} dy$ for all $x \in \mathbb{R}$, given Dirichlet conditions, etc.)

Notice how: $f(x) = \int_{-\infty}^{+\infty} e^{ixy} f(y) dy$. So, the fact that we know $\tilde{f}(f)(y)$ for almost every y doesn't matter.

⑨

Nyquist's theorem is very useful in signal processing.

It says that, if the Fourier transform of a signal is supported on a bounded interval, then we can reconstruct the signal only by knowing countably many of its values.

Lecture 27

①

→ Now, let us get back to our earlier simple observation:

Let $f: \mathbb{R} \rightarrow \mathbb{C}$, with $f(x) = 0 \quad \forall x \notin (-\pi, \pi)$ (then, $f(\pi) = f(-\pi) = 0$)

Then, if $f \in L^1(-\pi, \pi)$, is continuous

when extended 2π -periodically on the whole of \mathbb{R}

and satisfies (DC), then

$$f(x) = \sum_{k \in \mathbb{Z}} F(f)(k) \cdot e^{ikx}, \quad \text{for } x \in [-\pi, \pi]$$

Questions:

① What is $\sum_{k \in \mathbb{Z}} F(f)(k) e^{ikx}$ equal to, for the rest x in \mathbb{R} ?

② What is $\sum_{k \in \mathbb{Z}} F(f)(k) e^{ikx}$ equal to, when f doesn't vanish necessarily outside $[-\pi, \pi]$?

Answers: ① is easy. $\sum_{k \in \mathbb{Z}} F(f)(k) e^{ikx}$ is 2π -periodic,

so, given that we know what it equals to at each $x \in [-\pi, \pi]$, we also know its values on all $x \in \mathbb{R}$.

In particular, for general $x \in \mathbb{R}$, $\sum_{k \in \mathbb{Z}} F(f)(k) e^{ikx} = f(x + 2\pi m)$,

(2)

where m is the integer such that $x+2\pi m$ falls in $[-\pi, \pi]$.

Notice that $f(x+2\pi m)$ is just a horizontal translation of f , by $2\pi m$.

If we hadn't imposed $f(-n) = f(n) = 0$, then this way we would be doubling $f(n)$ and $f(-n)$.

And, since f is 0 outside $(-\pi, \pi)$, each such translation gives a function f with disjoint support.

In other words, $f(x+2\pi m) = \sum_{n=-\infty}^{+\infty} f(x+2\pi n)$.

That is, $\forall x \in \mathbb{R}, \sum_{n=-\infty}^{+\infty} f(x+2\pi n) = \text{the value of } f \text{ at the translation of } x, \text{ by multiples of } 2\pi, \text{ inside } [-\pi, \pi]$

$$\text{So, } \sum_{k \in \mathbb{Z}} F(f)(k) e^{ikx} = \sum_{n=-\infty}^{+\infty} f(x+2\pi n), \quad \forall x \in \mathbb{R}$$

superposition of shifts of f

The answer for ② is that the above holds for a much more general class of functions f than just those that vanish outside $(-\pi, \pi)$.

These are the functions of sub-polynomial growth, and are known as Schwartz functions. You can imagine

them as the smooth functions that decrease very rapidly to 0 after a while (but it could be a long

(3)

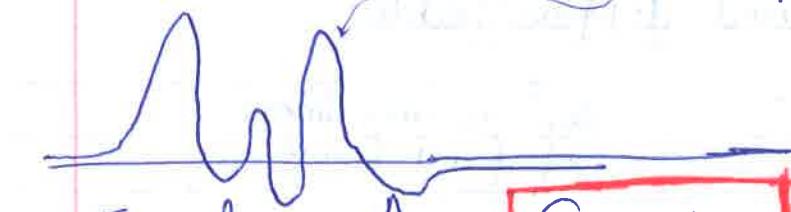
while, plus they don't have to ever vanish).

Here is the formal definition, if you are interested:

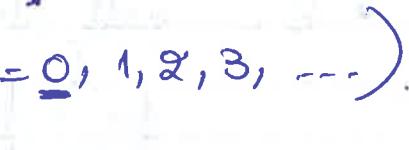
→ Def The set S of Schwartz functions is the set of functions $f: \mathbb{R} \rightarrow \mathbb{C}$, such that:

- f smooth (i.e. all its derivatives exist)
- f and its derivatives eventually grow slower than any polynomial

(i.e. $|f^{(k)}(x)| = O\left(\frac{1}{|x|^N}\right)$, for any $k, N = 0, 1, 2, 3, \dots$).



→ Example: Any Gaussian (i.e. any $f: \mathbb{R} \rightarrow \mathbb{C}$ with $f(x) = a e^{-\frac{(x-b)^2}{2c^2}}$)



is a Schwartz function

→ Observation: If $f \in S$, then, thanks to its fast decay, and the fact that it is smooth, $\sum_{n=0}^{+\infty} f(x+2\pi n)$ is well-defined and differentiable.

(4)

Poisson summation formula:

Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be a Schwartz function.

Then, $\sum_{k=-\infty}^{+\infty} F(f)(k) e^{ikx} = \sum f(x+2\pi n), \forall x \in \mathbb{R}.$

superposition of shifts of f .

Proof:

Let $h: \mathbb{R} \rightarrow \mathbb{C}$, with

$$h(x) = \sum_{n=-\infty}^{+\infty} f(x+2\pi n), \forall x \in \mathbb{R}.$$

- h is 2π periodic (this would be true even if f wasn't Schwartz).

- h is well-defined and differentiable.

due to fast decay of f

due to smoothness of f and fast decay.

Since h is 2π -periodic and differentiable, it equals its Fourier series everywhere (you can remember this; not as general as Dirichlet conditions, but easier).

So,
$$h(x) = \sum_{k \in \mathbb{Z}} \hat{h}(k) e^{ikx} \quad \forall x \in \mathbb{R},$$

where $\hat{f}(k) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} e^{-ikx} f(x) dx$ $\frac{e^{-ikx}}{e^{-ikx}} \frac{2\pi - \text{periodic}}{2\pi - \text{periodic}}$

$$= \frac{1}{2\pi} \cdot \int_0^{2\pi} e^{-ikx} \cdot \sum_{n=-\infty}^{+\infty} f(x+2\pi n) dx =$$

$$= \frac{1}{2\pi} \cdot \sum_{n=-\infty}^{+\infty} \int_0^{2\pi} e^{-ikx} f(x+2\pi n) dx$$

$x+2\pi n = y$
 $dx = dy$
 $x \in [0, 2\pi]$
 $\Rightarrow y \in [2\pi n, 2\pi(n+1)]$
 $e^{-ik(x+2\pi n)} = e^{-ikx}$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \int_{2\pi n}^{2\pi(n+1)} e^{-iky} f(y) dy$$

$$= \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{-iky} f(y) dy = \tilde{F}(f)(k), \quad \forall k \in \mathbb{Z}.$$

So,

$$\sum_{n=-\infty}^{+\infty} f(x+2\pi n) = \sum_{k=-\infty}^{+\infty} \tilde{F}(f)(k) e^{ikx}, \quad \forall x \in \mathbb{R}.$$



The above usually appears in literature as :

$$\sum_{n=-\infty}^{+\infty} f(x+2\pi n) = \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{ikx}, \quad \forall x \in \mathbb{R}$$

Remember, in this case \hat{f} has to be the continuous Fourier transform. I.e., $\hat{f}(k) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx$, not $\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx$.

(6)

→ Corollary: Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be a Schwartz function.

Then, $\sum_{n=-\infty}^{+\infty} f(2\pi n) = \sum_{k=-\infty}^{+\infty} F(f)(k)$

Proof: Set $x=0$ in Poisson's summation formula. ■

⚠ You can see Problem 22, p. 389 of the textbook, for an application of this that is needed in the theory of scattering of light.

→ The uncertainty principle:

The uncertainty principle is a statement that encapsulates the following heuristic claim:

If f is supported on an interval of length $\sim \varepsilon$,

then $F(f)$ is "large"
(not negligible) on $[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$;

and, similarly, $f(k)$ is not negligible for all $k \in [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$.

(7)

This is quite easy to believe, without a rigorous proof.

- Indeed, imagine $f: \mathbb{R} \rightarrow \mathbb{C}$ imagine $\varepsilon = \frac{1}{10^{10}}$, fixed.
 $f(x) = \begin{cases} 1, & x \in [-\varepsilon, \varepsilon] \\ 0, & \text{otherwise.} \end{cases}$

Looking at $f: [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}$, we see that, the \mathbb{Z} :

$$\hat{f}(k) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} e^{-ikx} f(x) dx = \frac{1}{2\pi} \cdot \int_{-\varepsilon}^{\varepsilon} e^{-ikx} dx.$$

And, since x runs in the tiny interval $[-\varepsilon, \varepsilon]$,

then kx runs in $[-k\varepsilon, k\varepsilon]$.

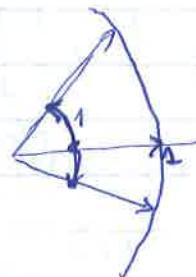
So, even when $|k|$ is quite large,

in fact whenever $k \in [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$,

then kx runs in $[-\frac{1}{\varepsilon} \cdot \varepsilon, \frac{1}{\varepsilon} \cdot \varepsilon] = [-1, 1]$; and these angles are small! So, the vector

e^{-ikx} , which we are integrating,

runs in this spectrum:



(8)

The average of these vectors is quite a large vector, as these vectors all point to pretty much the "same" direction.

So, $\frac{1}{2\epsilon} \cdot \int_{-\epsilon}^{\epsilon} e^{-ikx} dx$ is a "large" vector, ("pretty much" the vector \downarrow),

so $|\hat{f}(k)|$ is at least a constant multiple of ϵ , for all $k \in [-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$.

- Similarly, imagine $f(x) = \begin{cases} 1, & x \in [a-\epsilon, a+\epsilon] \\ 0, & \text{otherwise.} \end{cases}$, $a \in (n, n)$.

Then, $\forall k \in \mathbb{Z}$:

$$\hat{f}(k) = \frac{1}{2n} \cdot \int_{-n}^n e^{-ikx} f(x) dx = \frac{1}{2n} \cdot \int_{a-\epsilon}^{a+\epsilon} e^{-ikx} f(x) dx =$$

$x \in [a-\epsilon, a+\epsilon]$
 $\Rightarrow x-a \in [-\epsilon, \epsilon]$

$$= \frac{1}{2n} \int_{-\epsilon}^{\epsilon} e^{-iky} e^{-ika} dx$$

a fixed vector

When $k \in [-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$, then $k\epsilon \in [-1, 1]$, so e^{-iky} is

a tiny rotation, so the integrand is

pretty much e^{-ika} , $\forall x \in [-\epsilon, \epsilon]$. So, $\frac{1}{2n} \cdot \int_{-\epsilon}^{\epsilon} e^{-iky} e^{-ika} dx =$

(3)

$$= \frac{2E}{2\pi} \cdot \left[\frac{1}{2E} \cdot \int_{-\varepsilon}^{\varepsilon} e^{-iky} e^{-i\alpha y} dy \right] \sim (\text{a constant}) \cdot e^{-i\alpha y} \cdot \varepsilon,$$

pretty much $e^{-i\alpha y}$
fixed

so $|\hat{f}(k)| \geq (\text{a constant}) \cdot \varepsilon$.
(like $\frac{1}{2} \varepsilon$).



We are NOT claiming that $|\hat{f}(k)|$ is huge (after all, ε is small).
 But all $k \in [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$ contribute something non-negligible.
 As ε gets smaller, $[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$ gets bigger, and more k 's
 contribute something non-negligible (a multiple of ε).

- Similarly, if $f(x) = \begin{cases} 1, & x \in [\alpha - \varepsilon, \alpha + \varepsilon] \\ 0, & \text{otherwise} \end{cases}$

then, $\forall j \in \mathbb{R}, \exists \varepsilon \in [-\frac{1}{e}, \frac{1}{e}]$:

$$\hat{f}(j) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{-ijx} f(x) dx = \frac{1}{2\pi} \int_{\alpha - \varepsilon}^{\alpha + \varepsilon} e^{-ijx} dx \quad \frac{y=x-\alpha}{y \text{ runs in } [-\varepsilon, \varepsilon]}$$

$$= \frac{1}{2\pi} \cdot \int_{-\varepsilon}^{\varepsilon} e^{-ijy} e^{-iay} dy. \quad \Rightarrow y \text{ runs in } [-1, 1],$$

fixed

so e^{-ify} is a tiny rotation, so

$$\hat{f}(j) = \frac{2E}{2\pi} \cdot \left[\frac{1}{2E} \cdot \int_{-\varepsilon}^{\varepsilon} e^{-ify} e^{-iay} dy \right],$$

pretty much e^{-iay}
fixed unit vector

(10)

$$\text{so } |\hat{f}(g)| \geq (\text{constant}) \cdot \epsilon. \\ (\text{like } \frac{1}{2} \epsilon).$$

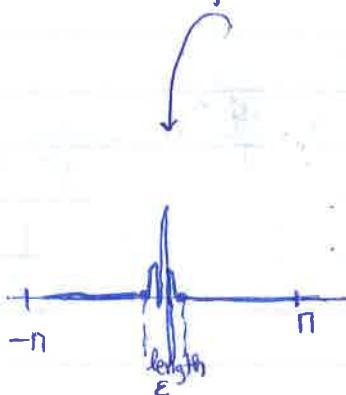
$\forall g \in [-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$.

Let us see this from another perspective

Suppose we know that f is localized on a tiny interval $[a-\epsilon, a+\epsilon]$.

Suppose we can write it as a Fourier series:

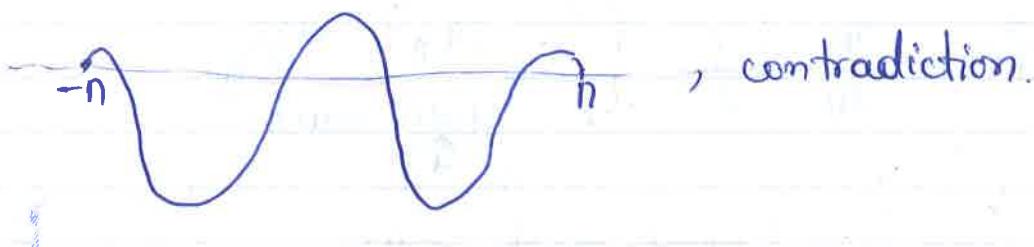
$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}, \quad \forall x \in [a-\epsilon, a+\epsilon]$$



Suppose that there was only 1 dominant coefficient, and the rest were negligible.

Then, f would practically be an exponential, so it would look like

this:



It makes sense that many exponentials should be required to get such a localised sum as f . And the uncertainty principle tells us, morally, exactly that: many Fourier coefficients contribute to f , in the case where f is localised. (sort of "uniformly")

A nigorous statement of the uncertainty principle, if you are interested, says the following:

The uncertainty principle:

Let $f \in L^2(\mathbb{R})$. Suppose that $f = 0$ outside an interval of length r (small).

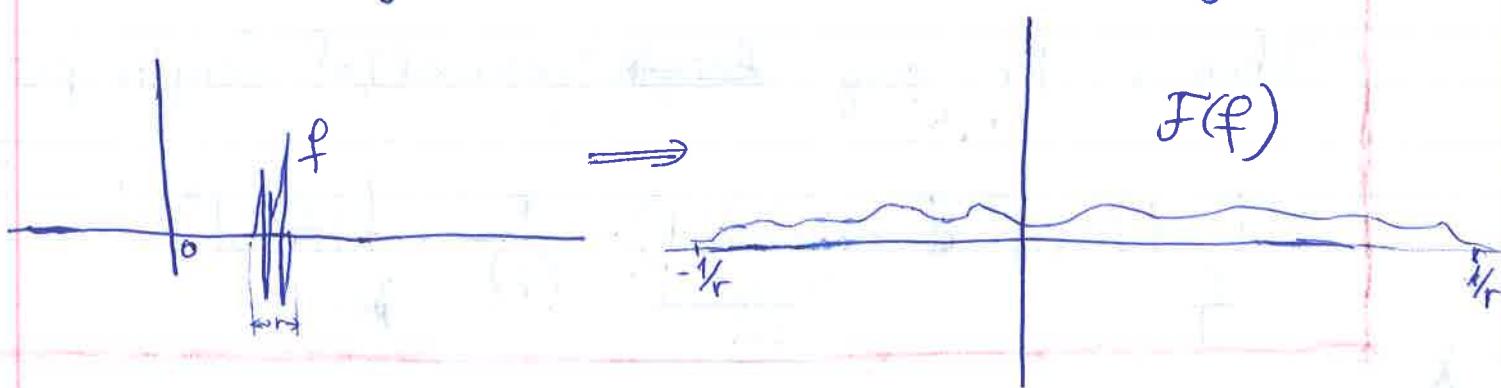
Then, for any interval I_p , of length $p \leq \frac{1}{r}$,

$$\int_{I_p} |f'(f)|^2 \leq (\text{constant}) \cdot \underbrace{\frac{p}{(\frac{1}{r})}}_{\text{independent of } I_p} \cdot \int_{\mathbb{R}} |f'(f)|^2.$$

! This means that the energy of $f'(f)$ on I_p is just a proportion of whole energy. And the proportion is $\frac{\text{length } p}{\text{length } \frac{1}{r}}$. I.e., it is as if $f'(f)$ is somehow

"uniformly distributed" along an interval $I_{1/r}$ of length $1/r$, so that the energy on each small part I_p of $I_{1/r}$ is at most a proportion of I_p in $I_{1/r}$. There is no "clustering" of energy.

Remember, the uncertainty principle has to do with how many frequencies contribute something, NOT with proving that there exist frequencies contributing a lot. Potentially no frequency contributes a lot; this is really what the above statement says.



actually, there is decay $F(f)(t) \xrightarrow{|t| \rightarrow \infty} 0$, but it isn't necessary that this decay truly starts at distance $\frac{1}{r}$ from 0. It may start later.